COMPUTATIONAL ONTOLOGIES OF PARTHOOD, COMPONENTHOOD, AND CONTAINMENT

Logical properties of foundational relations: a (non-exhaustive) comparison between FOL and DL

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AIM

- Provide an insight on how we can specify the semantics of foundational relations using different types of logics (deductive systems): FOL and DL

- Taken From:

    - Thomas Bittner, Maureen Donnelly: Logical properties of foundational relations in bio-ontologies. Artificial Intelligence in Medicine 39(3): 197-216 (2007)

    - Thomas Bittner, Maureen Donnelly: Computational ontologies of parthood, componenthood, and containment. IJCAI 2005: 382-387

AIM OF BITTNER & DONNELLY

- *Parthood*, *componenthood*, and *containment* relations are commonly assumed in biomedical ontologies and terminology systems, but are not usually clearly distinguished from another.
- clarify *distinctions* between these relation as well as principles governing their *interrelations;*
- develop a theory of these relations in first order predicate logic and then discuss how description logics can be used to capture some important aspects of the first order theory.

MY AIM

- Exemplify how to specify the semantics of foundational relations (*parthood*, *componenthood*, and *containment*) using different types of formal deductive systems: first-order logic (FOL) and description logics (DLs).

- I do not aim at discussing the adequacy of the ontological analysis of *parthood*, *componenthood*, and *containment* presented in the paper.

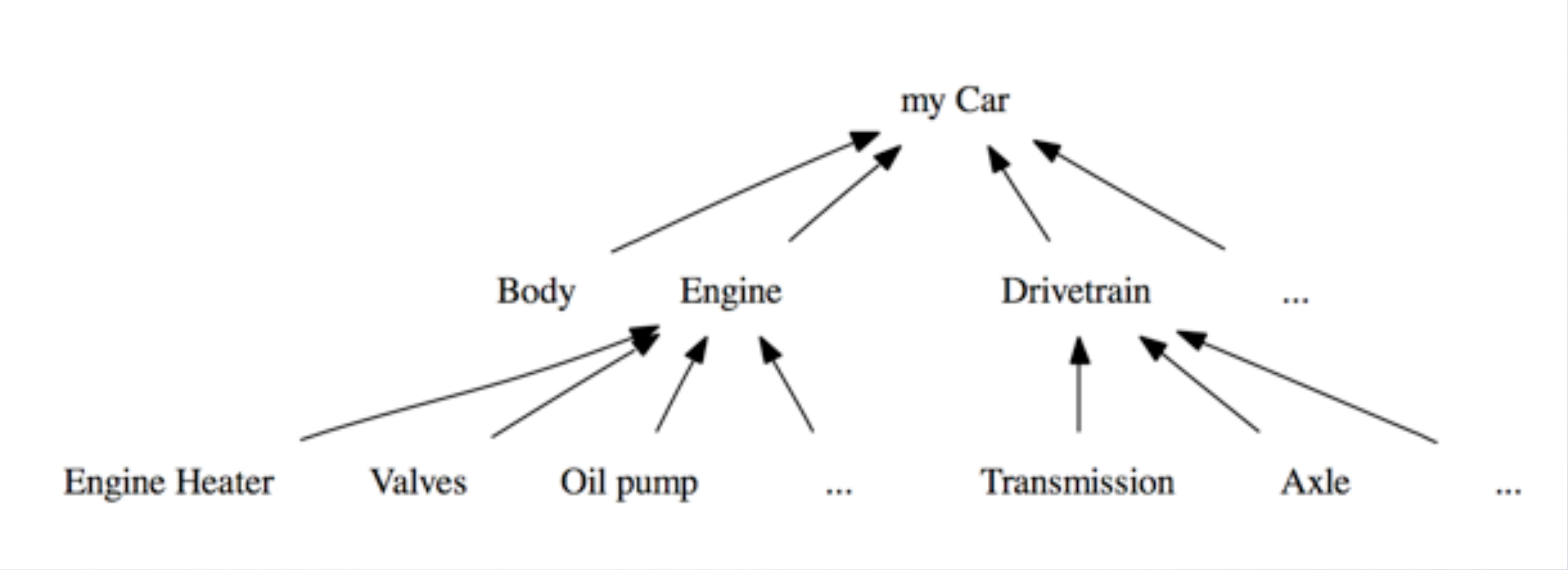
(PROPER) PARTHOOD

- Intuitively, proper parthood relations determine the general part-whole structure of an object.
- The left side of my car is a proper part-of my car
- The upper part of my body is a proper part-of my body

COMPONENTHOOD

- Intuitively, a component of an object is a proper part of that object which has a complete bona fide boundary (i.e., boundary that correspond discontinuities in reality) and a distinct function.

- My car has components, for example, its engine, its oil pump, its wheels, etc.



CONTAINMENT

- Intuitively containment is here understood as a relation which holds between disjoint material objects when one object (the containee) is located within a space partly or wholly enclosed by the container.
- My car is a container. It contains the driver in the seat area and a tool box and a spare-tire in its trunk.

RELATED RELATIONS

- All components of my car are parts of my car, but my car has also parts (e.g., its left part) that are not components.

Being a component implies being a part of

(Being a part of does not imply being a component)

- If the left side of my tool box is proper part of my toolbox and the toolbox is contained in the boot of my car, then the left side of my toolbox is contained in my car.

If x is proper part of y and y is contained in z then x is contained in z

SIMILAR ASPECTS

- All three relations are transitive and asymmetric.
- Examples of containment:
  - The screw-driver is contained in my tool box and the tool box is contained in the trunk of my car, therefore the screw-driver is contained in the trunk of my car.
  - If the tool box is contained in the trunk of my car, then the trunk of my car is not contained in the tool box.

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  - If the tool box is contained in the trunk of my car, then the trunk of my car is not contained in the tool box.
- Since they are similar they are not always clearly distinguished in bio-medical ontologies such as GALEN or SNOMED.

DIFFERENT ASPECTS

- There can be a container with a single containee (e.g., the screw-driver is the only tool in my tool box) but no object can have single proper part.

- Two components share a component only when one is a sub-component of the other. Instead, the left half of my car and the bottom half of my car share the bottom left part of my car but they are not proper parts of each other.

WHY A (FORMAL) ONTOLOGY?

- To make explicit the semantics of these three terms.
- By explicating the distinct properties of proper parthood, componenthood and containment relations.
- That is, to specify the meaning of terms such as ‘proper-part-of’, ‘component-of’, and ‘contained-in’.

THE WORK-PLAN

1. Characterize important properties of binary relations and see how they apply to *parthood*, *componenthood*, and *containment*;

2. Use these properties to provide a formal theory of *parthood*, *componenthood*, and *containment* in FOL;

3. Study how to formulated the 'same' theory in description logic.

PRELIMINARIES

- R-structure (Δ, R) consists of a non-empty domain Δ and a non-empty binary relation $R \subseteq (\Delta \times \Delta)$
- $R(x, y)$ indicates that *R* holds between *x* and *y.*
- We introduce 3 relations based on *R*:

$$R_{=}(x, y) =_{df} R(x, y) \textit{ or } x = y$$
$$R_{O}(x, y) =_{df} \exists z \in \Delta : \; R_{=}(z, x) \;\&\; R_{=}(z, y)$$
$$R_{i}(x, y) =_{df} R(x, y) \& (\neg \exists z \in \Delta : \; R(x, z) \;\&\; R(z, y))$$

- Note: For a given R-structure, the three relations may be empty or identical to R.

PROPERTIES OF BINARY RELATIONS

property	description
reflexive	$\forall x \in \Delta : R(x,x)$
irreflexive	$\forall x \in \Delta$: not $R(x,x)$
symmetric	$\forall x, y \in \Delta$: if $R(x,y)$ then $R(y,x)$
asymmetric	$\forall x, y \in \Delta$: if $R(x,y)$ then not $R(y,x)$
transitive	$\forall x, y, z \in \Delta$: if $R(x,y)$ and $R(y,z)$ then $R(x,z)$
intransitive	$\forall x, y \in \Delta$: if $R(x,y)$ and $R(y,z)$ then not $R(x,z)$

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- The identity relation is reflexive, symmetric and transitive;
- R_O is symmetric, R_i is intransitive, and $R_=$ is reflexive;
- On their respective domains proper parthood, componenthood, and containment are asymmetric and transitive;

PROPERTIES OF BINARY RELATIONS

property	description
up-discrete	$\forall x, y \in \Delta$: if $R(x, y)$ then $R_i(x, y)$ or $\exists z \in \Delta$: $R(x, z)$ and $R_i(z, y)$
dn-discrete	$\forall x, y \in \Delta$: if $R(x, y)$ then $R_i(x, y)$ or $\exists z \in \Delta$: $R_i(x, z)$ and $R(z, y)$
discrete	up-discrete & dn-discrete
dense	$\forall x, y \in \Delta$: if $R(x, y)$ then $\exists z \in \Delta$: $R(x, z)$ and $R(z, y)$

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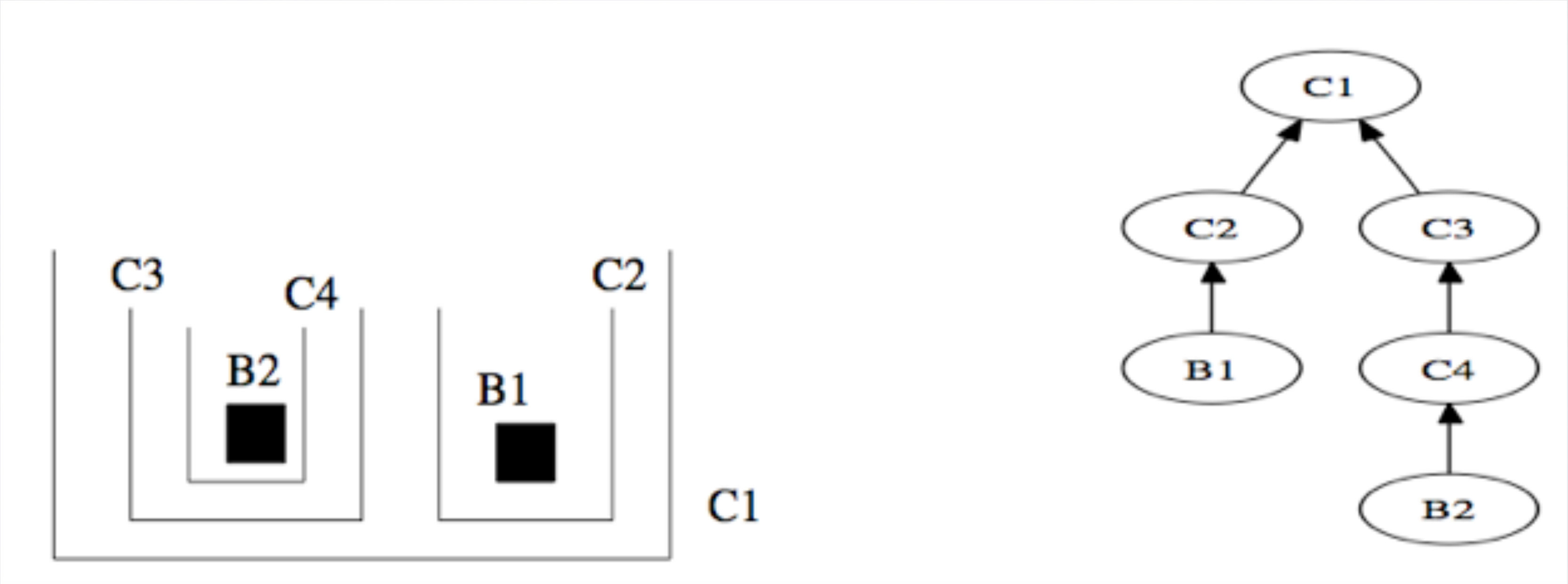
discrete	up-discrete & dn-discrete
dense	$\forall x, y \in \Delta$: if $R(x, y)$ then $\exists z \in \Delta$: $R(x, z)$ and $R(z, y)$

- Containment is discrete (**contained-in**);
- Componenthood is discrete (**component-of**);
- Proper-parthood is dense (**proper-part-of**);

CONTAINMENT IS DISCRETE



$$\Delta_C = \{C_1, C_2, C_3, C_4, B_1, B_2\}$$

- if *x* is contained-in *y* then either

  (a) *x* is immediately contained in *y* or

  (b) there exists a *z* such that *x* is an immediately contained in *z* and *z* is contained in *y*, or

  (c) there exists a *z* such that *x* is contained in *z* and *z* is immediately contained in *y*.

- Similarly for componenthood.

PROPER PARTHOOD IS DENSE

- Due to the existence of fiat parts (parts which lack a complete bona fide boundary).
- Consider my car and its proper parts. My car does not have an immediate proper part – Whatever proper part *x* we chose, there exists another slightly bigger proper part of my car that has *x* as a proper part.

PROPERTIES OF BINARY RELATIONS

property	description
WSP	$\forall x, y \in \Delta$: if $R(x, y)$ then $\exists z \in \Delta$: $R(z, y)$ & not $R_O(z, x)$
NPO	$\forall x, y \in \Delta$: if $R_O(x, y)$ then $x = y$ or $R(x, y)$ or $R(y, x)$
NSIP	$\forall x, y \in \Delta$: if $R_i(x, y)$ then $\exists z \in \Delta$: $R_i(z, y)$ & not $x = z$
SIS	$\forall x, y, z \in \Delta$: if $R_i(x, y)$ and $R_i(x, z)$ then $y = z$

- WSP = weak supplementation property;
- NPO = no-partial-overlap;
- NSIP = no-single-immediate-predecessor;
- SIS = single-immediate-successor.

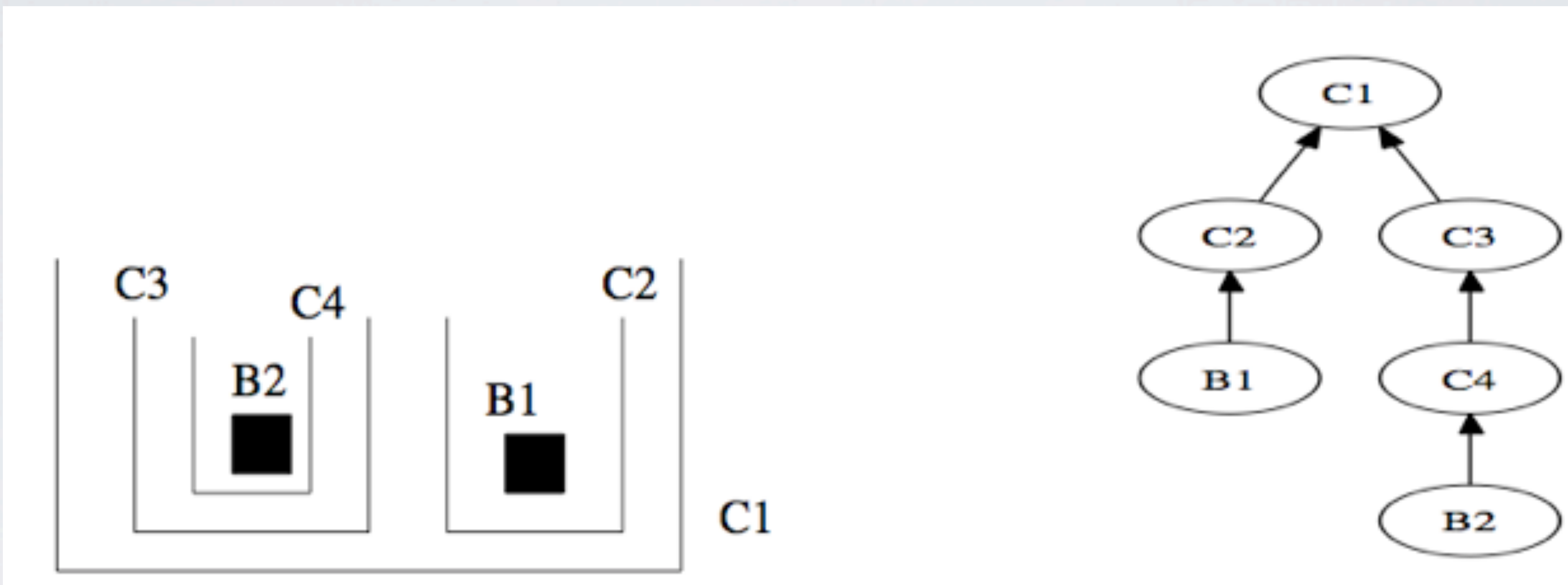
WEAK SUPPLEMENTATION PROPERTY

- **proper-part-of** is proper parthood on the domain of spatial objects;
- **proper-part-of**$_o$ is the overlap relation;
- WPS tell us tells us that if *x* is a proper part of *y* then there exists a proper part *z* of *y* that does not overlap *x*.
- Example: since the left side of my car is a proper part of my car there is some proper part of my car (e.g., the right side of my car) which does not overlap with the left side of my car.

WEAK SUPPLEMENTATION PROPERTY

- **component-of** is componenthood on the domain of artifacts;
- **proper-part-of$_o$** is the relation of sharing a component;
- WPS tell us tells us that if *x* is a component of *y* then there exists a component *z* of *y* such that z and x do not have a common component.
- Example: since the engine of my car is a component of my car there is some component of my car (e.g., the body of my car) which does not have a component in common with the engine.

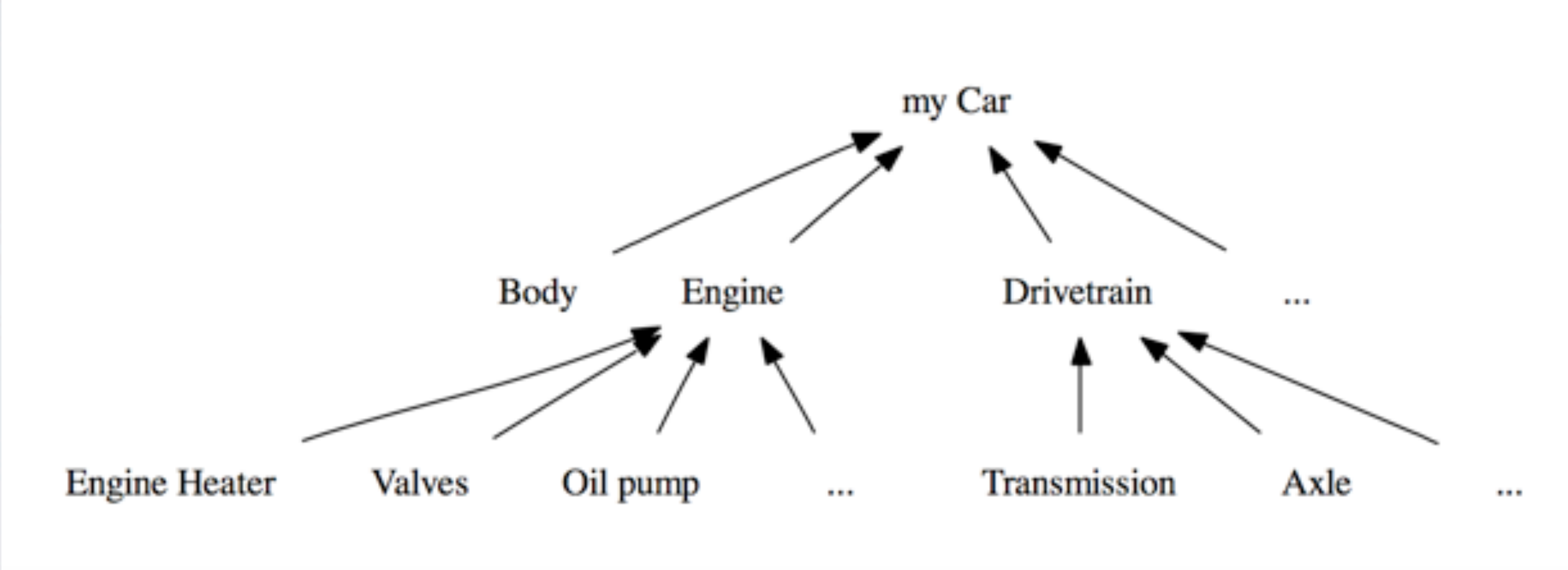
WEAK SUPPLEMENTATION PROPERTY



$$\Delta_C = \{C_1, C_2, C_3, C_4, B_1, B_2\}$$

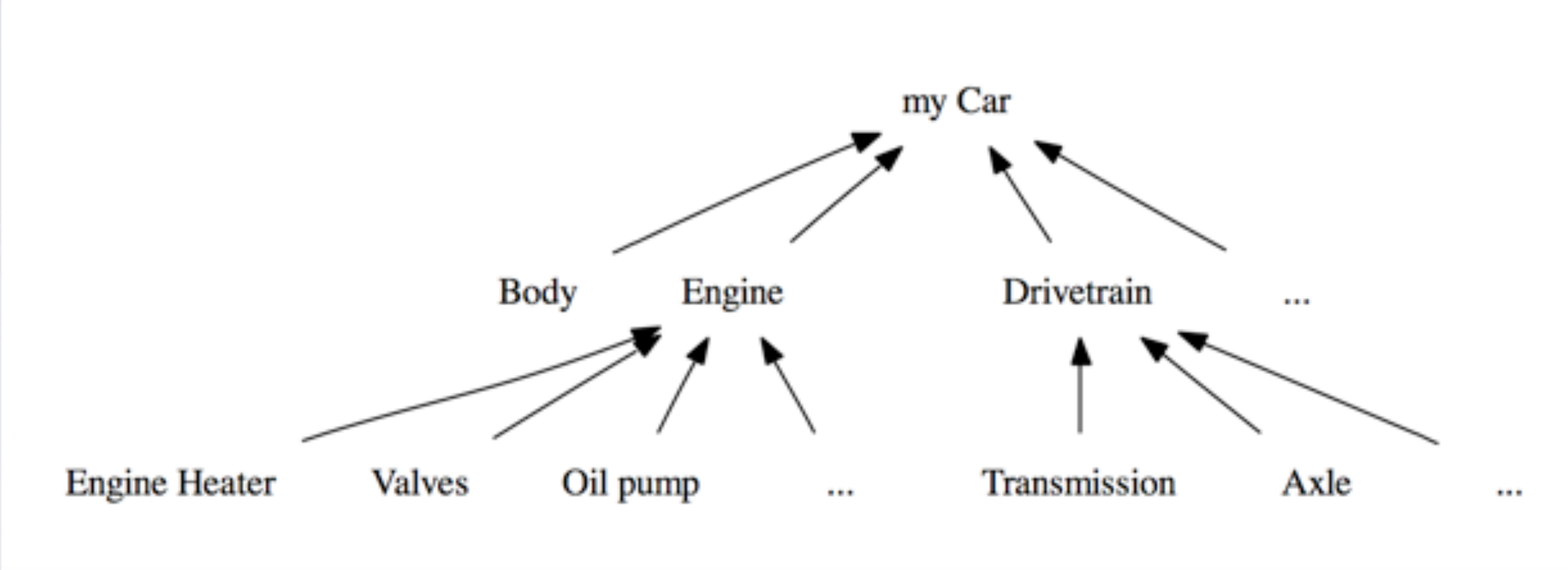
- **contained-in** defined over Δ_C does not satisfy WPS

NO-PARTIAL-OVERLAP



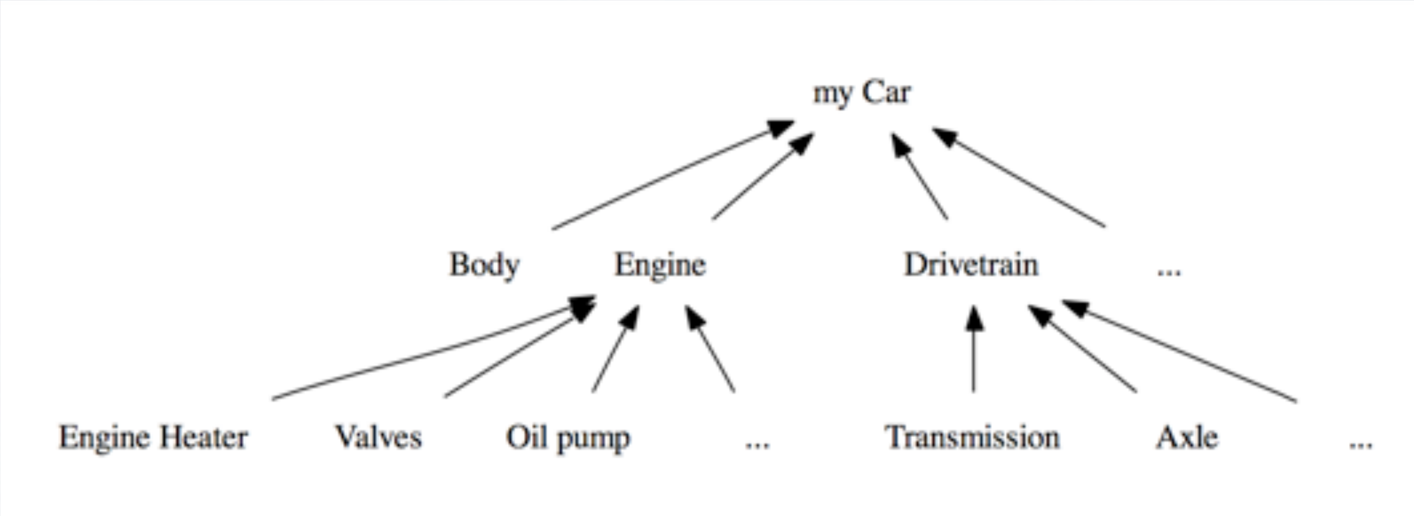
- **component-of** in the diagram satisfies the NPO property
- **proper-part-of** in the spatial domain does not have the NPO property (the left half and the bottom half of my car overlap partially).

SINGLE-IMMEDIATE-SUCCESSOR

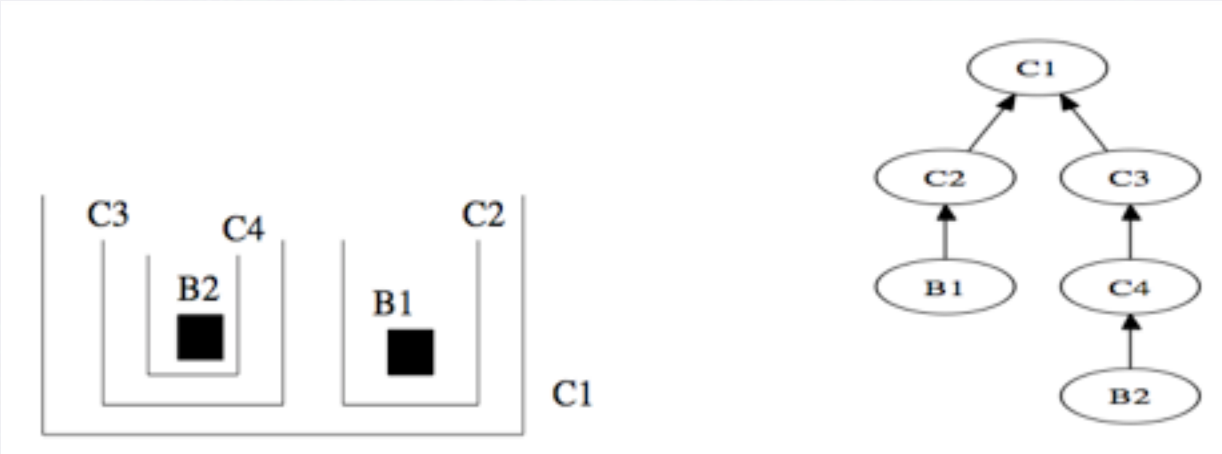


- **component-of** in the diagram satisfies the SIS property
- containment often does not have SIS: the tool box in the trunk of my car is also contained in my car. My car and the trunk of my car are distinct immediate containers for my tool box.

NO-SINGLE-IMMEDIATE-PREDECESSOR



- **component-of** in the diagram satisfies the NSIS property



- **contained-in** in the diagram does not have NSIS:

RELATIONS ABOUT THESE PROPERTIES

- NPO implies SIS;
- if R is finite and has the SIS then it has the NPO;
- if R is up-discrete and NPO then it also has the SWP iff it has the NSIP;
- if R is reflexive then R_i is empty.

USEFUL R-STRUCTURES

R-structure	properties
Partial Ordering (PO)	asymmetric, transitive
Discrete PO	PO + discrete
Parthood structure	PO + WSP + dense
Component-of structure	PO + WSP + NPO + discrete

- $(\Delta, \mathbf{PP}, \mathbf{CntIN}, \mathbf{CmpOf})$ is a parthood-containment-component structure iff:
  1. $(\Delta, \mathbf{PP})$ is a parthood structure;
  2. $(\Delta, \mathbf{CntIn})$ is a discrete PO;
  3. $(\Delta, \mathbf{CmpOf})$ is a component-of structure;
  4. if $\mathbf{CntIn}(x, y)$ and $\mathbf{PP}(y, z)$ then $\mathbf{CntIn}(x, z)$
  5. if $\mathbf{PP}(x, y)$ and $\mathbf{CntIn}(y, z)$ then $\mathbf{CntIn}(x, z)$
  6. if $\mathbf{CmpOf}(x, y)$ then $\mathbf{PP}(x, y)$

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  6. if $\mathbf{CmpOf}(x, y)$ then $\mathbf{PP}(x, y)$

But, where is logic?
Until now we have performed our analysis using the language of mathematics (that is, identifying the structures defined by our relations.

FORMALISING IN LOGIC

- First order logic with equality (identity);
- $PP, CntIn,$ and $CmpOf$ are three predicates whose intended interpretation are the relations $\mathbf{PP}, \mathbf{CntIn},$ and $\mathbf{CmpOf}$ of a parthood-containment-component structure.

AXIOMS FOR *PP*

$$D_{PP_=} \quad PP_= \, xy \equiv PP \, xy \vee x = y$$
$$D_{PP_O} \quad PP_O \, xy \equiv (\exists z)(PP_= \, zx \wedge PP_= \, zy)$$

$APP1 \quad PP \, xy \rightarrow \neg PP \, yx$ (*Asimmetry*)

$APP2 \quad (PP \, xy \wedge PP \, yz) \rightarrow PP \, xz$ (*Transitivity*)

$APP3 \quad PP \, xy \rightarrow (\exists z)(PP \, zy \wedge \neg PP_O \, zx)$ ***(WSP)***

$APP4 \quad PP \, xy \rightarrow (\exists z)(PP \, xz \wedge PP \, zy)$ (*Density*)

- Models of the Theory which contains APP1-4 are the parthood structures.

AXIOMS FOR *CMPOF*

$D_{\mathit{CmpOf}_=}$ $\quad \mathit{CmpOf}_=\, xy \equiv \mathit{CmpOf}\, xy \vee x = y$

D_{CmpOf_O} $\quad \mathit{CmpOf}_O\, xy \equiv (\exists z)(\mathit{CmpOf}_=\, zx \wedge \mathit{CmpOf}_=\, zy)$

$ACP1$ $\quad (\mathit{CmpOf}\, xy \wedge \mathit{CmpOf}\, yz) \rightarrow \mathit{CmpOf}\, xz$ *(Transitivity)*

$ACP2$ $\quad \mathit{CmpOf}\, xy \rightarrow PP\, xy$ *(Property 6 of PCC structures)*

$TCP1$ $\quad \mathit{CmpOf}\, xy \rightarrow \neg \mathit{CmpOf}\, yx$ *(Asymmetry as a Theorem!!)*

AXIOMS FOR *CMPOF*

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$$D_{CmpOf_O} \quad CmpOf_O\, xy \equiv (\exists z)(CmpOf_=\, zx \wedge CmpOf_=\, zy)$$

$$ACP1 \quad (CmpOf\, xy \wedge CmpOf\, yz) \rightarrow CmpOf\, xz$$ (Transitivity)

$$ACP2 \quad CmpOf\, xy \rightarrow PP\, xy$$ (Property 6 of PCC structures)

$$TCP1 \quad CmpOf\, xy \rightarrow \neg CmpOf\, yx$$ (Asymmetry as a Theorem!!)

$$D_{CmpOf_i} \quad CmpOf_i\, xy \equiv CmpOf\, xy \wedge \neg(\exists z)(CmpOf\, xz \wedge CmpOf\, zy)$$

$$ACP3 \quad CmpOf\, xy \rightarrow (CmpOf_i\, xy \vee ((\exists z)(CmpOf_i\, xz \wedge CmpOf\, zy) \wedge (\exists z)(CmpOf\, xz \wedge CmpOf_i\, zy)))$$

AXIOMS FOR *CMPOF*

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D_{CmpOf_O} $\quad CmpOf_O \, xy \equiv (\exists z)(CmpOf_= \, zx \wedge CmpOf_= \, zy)$

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$TCP2$ $\quad CmpOf_i \, xy \wedge CmpOf_i \, yz \rightarrow \neg CmpOf_i \, xz$

AXIOMS FOR *CMPOF*

$D_{CmpOf_=}$ $\quad CmpOf_= \, xy \equiv CmpOf \, xy \vee x = y$

D_{CmpOf_O} $\quad CmpOf_O \, xy \equiv (\exists z)(CmpOf_= \, zx \wedge CmpOf_= \, zy)$

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$TCP2$ $\quad CmpOf_i \, xy \wedge CmpOf_i \, yz \rightarrow \neg CmpOf_i \, xz$

(Transitivity as a Theorem!!)

AXIOMS FOR *CMPOF*

$$ACP4 \quad \textit{CmpOf}_O xy \rightarrow (\textit{CmpOf}_= xy \vee \textit{CmpOf}\, zx)$$ (*NPO*)

$$ACP5 \quad \textit{CmpOf}_i\, xy \rightarrow (\exists z)(\textit{CmpOf}_i\, zy \wedge \neg z = x)$$ (*NSIP*)

$$TCP3 \quad \textit{CmpOf}\, xy \rightarrow (\exists z)(\textit{CmpOf}\, zy \wedge \neg \textit{CmpOf}_O\, zx)$$ (*WSP*)

$$TCP4 \quad \textit{CmpOf}_i\, xz_1 \wedge \textit{CmpOf}_i\, xz_2 \rightarrow z_1 = z_2$$ (*No two distinct immediate successors*)

- Models of the Theory which contains ACP1-5 are component-of structures.

AXIOMS FOR *CNTIN*

$$D_{\mathit{CntIn}_=} \quad \mathit{CntIn}_=\, xy \equiv \mathit{CntIn}\, xy \vee x = y$$

$$D_{\mathit{CntIn}_O} \quad \mathit{CntIn}_O\, xy \equiv (\exists z)(\mathit{CntIn}_=\, zx \wedge \mathit{CntIn}_=\, zy)$$

$$D_{\mathit{CntIn}_i} \quad \mathit{CntIn}_i\, xy \equiv \mathit{CntIn}\, xy \wedge \neg(\exists z)(\mathit{CntIn}\, xz \wedge \mathit{CntIn}\, zy)$$

$$ACT1 \quad \mathit{CntIn}\, xy \rightarrow \neg \mathit{CntIn}\, yx$$ *(Asimmetry)*

$$ACT2 \quad (\mathit{CntIn}\, xy \wedge \mathit{CntIn}\, yz) \rightarrow \mathit{CntIn}\, xz$$ *(Transitivity)*

$$ACT3 \quad \mathit{CntIn}\, xy \rightarrow (\mathit{CntIn}_i\, xy \vee ((\exists z)(\mathit{CntIn}_i\, xz \wedge \mathit{CntIn}\, zy) \wedge (\exists z)(\mathit{CntIn}\, xz \wedge \mathit{CntIn}_i\, zy)))$$ *(Discreteness)*

$$ACT4 \quad \mathit{PP}\, xy \wedge \mathit{CntIn}\, yz \rightarrow \mathit{CntIn}\, xz$$

$$ACT5 \quad \mathit{CntIn}\, xy \wedge \mathit{PP}\, yz \rightarrow \mathit{CntIn}\, xz$$

(Property 4 and 5 of PCC structures)

THE FORMAL THEORY

- FO-PCC is the theory containing axioms APP1-4, ACP1-5 and ACT1-5;
- Parthood-containment-component structures are models of this theory;
- Via reasoning we can:
  - infer properties on data (e.g., using transitivity);
  - check constraints (e.g., check if all the data comply with asymmetry)
  - check equivalence in ontology integration (e.g., assume that another ontology has a symbol << in its terminology. Is this just a rewriting of *PP*? I.e. are the logical properties of these two predicates identical?)
- Reasoning is nice but reasoning in FOL is undecidable. So?

FO-PCC IN DL

$\mathcal{L}_{WSP}$

$$
\begin{aligned}
&\top^{\mathcal{I}} = \Delta, \bot^{\mathcal{I}} = \emptyset;\\
&(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}},\\
&(\sim C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}};\\
&(\exists R.C)^{\mathcal{I}} = \{a \in \Delta \mid (\exists b)((a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}})\}\\
&(\forall R.C)^{\mathcal{I}} = \{a \in \Delta \mid (b)((a,b) \in R^{\mathcal{I}} \rightarrow b \in C^{\mathcal{I}})\}\\
&(= 1R)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \mid (a,b) \in R^{\mathcal{I}}\}| = 1\}\\
&(S \sqcap T)^{\mathcal{I}} = S^{\mathcal{I}} \cap T^{\mathcal{I}}, (S \sqcup T)^{\mathcal{I}} = S^{\mathcal{I}} \cup T^{\mathcal{I}},\\
&(\sim S)^{\mathcal{I}} = \Delta \times \Delta \setminus S^{\mathcal{I}};\\
&(S \circ T)^{\mathcal{I}} = \{(a,c) \in \Delta \times \Delta \mid\\
&\qquad (\exists b)((a,b) \in S^{\mathcal{I}} \wedge (b,c) \in T^{\mathcal{I}})\}\\
&\mathtt{Id}^{\mathcal{I}} = \{(a,a) \mid a \in \Delta\}\\
&(\mathtt{R}^{-})^{\mathcal{I}} = \{(b,a) \in \Delta \times \Delta \mid (a,b) \in R^{\mathcal{I}}\}
\end{aligned}
$$

FO-PCC IN DL

$\mathcal{L}_{WSP}$

$\top^{\mathcal{I}} = \Delta, \bot^{\mathcal{I}} = \emptyset;$

$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}},$

$(\sim C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}};$

$(\exists R.C)^{\mathcal{I}} = \{a \in \Delta \mid (\exists b)((a, b) \in$

$(\forall R.C)^{\mathcal{I}} = \{a \in \Delta \mid (b)((a, b) \in$

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This is the language in which we can state FO-PCC (APP1-4, ACP1-5 and ACT1-5)!

ENCODING OF RELATIONS

(asym)	$R^{-} \sqsubseteq (\sim R)$
(trans)	$R \circ R \sqsubseteq R$
(intrans)	$R \circ R \sqsubseteq (\sim R)$
(NPO)	$(R^{-} \circ R) \sqsubseteq (R \sqcup Id \sqcup R^{-})$
(WSP)	$R^{-} \sqsubseteq R^{-} \circ \sim ((R^{-} \sqcup Id) \circ (R \sqcup Id))$
(dense)	$(R \sqcap (\sim Id)) \sqsubseteq R \circ R$
(def-i)	$R_i \doteq R \sqcap \sim (R \circ R)$
(discrete)	$R \sqsubseteq R_i \sqcup (R \circ R_i \sqcap R_i \circ R)$
(SIS)	$\exists R_i.\top \sqsubseteq (=1)R_i.\top$
(NSIP)	$(=1)R_i^{-}.\top \sqsubseteq \bot$

(Reflexive) $Id \sqsubseteq R$

(Irreflexive) $Id \sqsubseteq \sim R$

(Symmetric) $R^{-} \sqsubseteq R$

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(Symmetric) $R^- \sqsubseteq R$

Using this encoding we can "translate" in DL APP1-4, ACP1-5 and ACT1-5!

BUT, $\mathcal{L}_{WSP}$ is undecidable

AND SO...?

- It is important to identify less complex sub-languages of $\mathcal{L}_{WSP}$ that are still sufficient to state axioms distinguishing parthood, componenthood, and containment relations.

- Otherwise the DL version of FO-PCC would have no computational advantages over the first order theory.

THE LANGUAGE $\mathcal{L}$

$$\top^{\mathcal{I}} = \Delta, \bot^{\mathcal{I}} = \emptyset;$$
$$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}},$$
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Role composition only appears in acyclic role terminologies with expressions of the form

$\mathtt{R} \circ \mathtt{R} \sqsubseteq \mathtt{R}$

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$\mathcal{L}$ is decidable

but... what do we lose?

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THE LANGUAGE $\mathcal{L}$

- We can express:
  - transitivity
  - the relations between PP, $CntIn$, and $CmpOf$
- but no:
  - asymmetry, WSP property, NPO property, $\mathtt{R}_i$ in terms of $\mathtt{R}$, irreflexivity.
- We can express SIS and SISP but not that $\mathtt{R}_i$ is a sub-relation of $\mathtt{R}$

(SIS)	$\exists \mathbf{R}_\mathtt{i}.\top \sqsubseteq (=1)\mathbf{R}_\mathtt{i}.\top$
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but... we do lose too much!!

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ANYTHING IN THE MIDDLE?

$\mathcal{L}^{\sim \mathtt{Id} \sqcup}$

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Role negation is restricted to role names

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THE LANGUAGE $\mathcal{L}^{\sim \mathtt{Id} \sqcup}$

- We can also express:
  - irreflexivity, intransitivity, asymmetry, NPO
- but still no:
  - WSP property, $\mathtt{R}_i$ in terms of $\mathtt{R}$, irreflexivity.

- and, is $\mathcal{L}^{\sim \mathtt{Id} \sqcup}$ decidable? Open Problem.

CONCLUSIONS OF BITTNER & DONNELLY

- Formal properties of parthood, componenthood and containment relations investigated.
- first order logic has the expressive power required to distinguish important properties of these relations
- DLs seem not appropriate for formulating complex interrelations between relations.
- A way out. A computational ontology consists of two components:
  - a DL based ontology that enables automatic reasoning and constrains meaning as much as possible and
  - a FOL ontology that serves as meta-data and makes explicit properties of relations that cannot be expressed in computationally efficient DLs.