Introduction to first order logic for knowledge representation
Part 1 - Introduction to Logic

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What is Logic?

- The main objective of a logic (there is not a unique logic but many) is to express by means of a formal language the knowledge about a he certain phenomena or a certain portion of the world.

- encode with a precise set of deterministic rules, called inference rules, the basic reasoning steps which are considered to be correct by everybody.

- a correct reasoning allows to show that a certain knowledge is a logical consequence of a given set of facts, and in logic correct reasoning chains are constructed by concatenating applications of simple inference rules, that allow to transform the initial knowledge into the conclusion one wants to derive.

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Language, real world and mathematical struture

When we want to describe and reason about some phenomena of the real world, we use sentences of a Language, and, If we are also interested in a more regorous description of the phenomena, we also provide a matematical model that is an abstraction of the portion of the real world we are interested in.
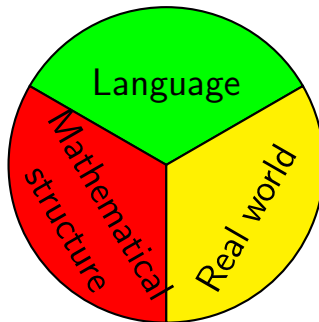
Language, real world and mathematical structure

Language

In describing a phenomena or a portion of the world, we adopt a language. The phrases of this language are used to describe objects of the real worlds, their properties, and facts that holds. This language can be *informal* (natural language, graphical language, icons, etc...) or a *formal* (logical language, programming language, mathematical language, ...) It is also possible to have mixed language, i.e. languages with parts which are formal, and other that are informal (e.g., class diagrams in UML)

Real world

Provide a coplete description of the real world is clearly impossible, and maybe also useless. However, typically one is interested on a portion of the world, e.g., a particular physical phenomena, or a social aspect, or modeling rationality of people, or simply logical circuits, ...
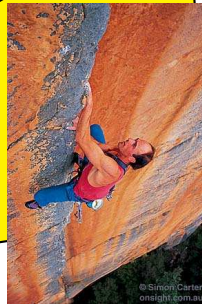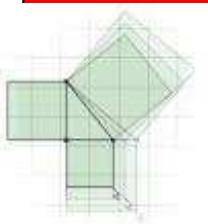
Mathematical model

The mathematical model constitute an abstraction of a portion of the world. It represents under the shape of mathematical objects, such that sets, relations, functions, ... real world entities. In everyday communication, we are not referring to such mathematical models, but, especially in science, in order to show that a certain argumentation is correct, people provide mathematical models that describes in an abstract and concise manner the specific aspect of the real world.

Facts about euclidean geometry can be expressed in terms of natural language, and they can refer to one or more real world situation. (in the picture it refers to the composition of the forces in free climbing). However, the importance of the theorem lays in the fact that it describes a general property that holds in many different situations. All these different situations can be abstracted in the mathematical structure which is the euclidean geometry. So indeed the sentence can be interpreted directly in the mathematical structure. In this example the language is informal but it has an interpretation in a mathematical structure

In any right triangle, the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides that meet at a right angle)

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In a triangle ABC, if $\widehat{BAC}$ is right, then $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$

This example is obtained by the previous one by taking a language which is "more formal". Indeed the language mixes informal statements e.g. "if...then..." "is right" with some formal notation $\widehat{BAC}$ is an unambiguous and compact way to denote an angle, and similarly $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$ is a rigorous description of an equation that holds between the lengths of the triangle sides.
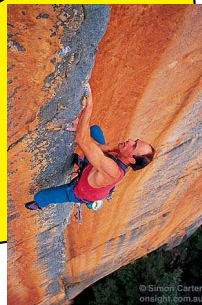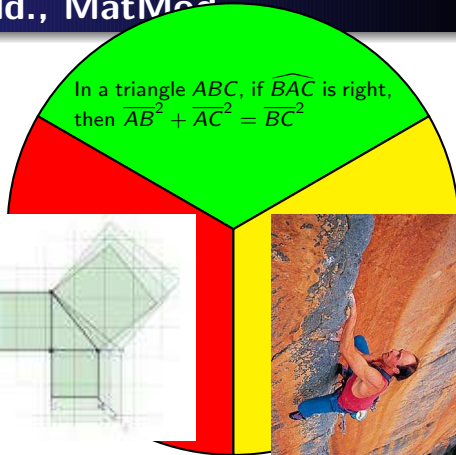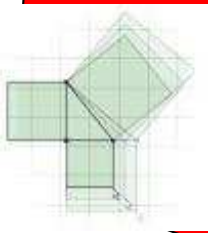
Connections between language, world and model

Intuitive interpretation (or informal semantics)

When you propose a new language (or when you have to learn a new language) it is important to associate to any element of the language an interpretation in the real world. This is called the intuitive interpretation (or informal semantics). For instance in learning a new programming language, you need to understand what is the effect in terms of execution of all the languages construct. For this reason the manual, typically, reports in natural language and with examples, the behavior of the language primitives. This is far to be a formal interpretation into a mathematical model. Therefore it is an informal interpretation

Formal interpretation (or formal semantics)

The formal semantics of a language is a function that allow to transform the elements of the language, as symbols, words, complex sentences, . . . into one or more elements of the mathematical structure. It is indeed the fromalization of the intuitive interpretation (or the intuitive semantics)

Abstraction

Is the link that connects the real world with it's matematical and abstract representation into a mathematical structure. If a certain situation is supposed to be abstractly described by a given structure, then the abstraction connects the elements that participats to the situation, with the components of the mathematical structure, and the properties that holds in the situation with the mathematical properties that holds in the structure.

Logic is a special case of the framework we have seen before, where the following important component are defined

- The language is a Logical language
- The formal interpretation allows to define a notion of truth
- It is possibel to define a notion of logical consequence between formulas I.e., if a set of formulas Γ are true then also ϕ is true

Formal language

Given a non empty set Σ of symbols called alphabet a formal language is a subset $L \subseteq \Sigma^*$, i.e., a set of finite strings of symbols in Σ. The elements of L are called well formed phrases. Formal languages can be specified by means of a grammar, i.e., a set of formation rules that allow to build complex well formed phrases. starting from simpler ones.

Logical language

Logical languages

A language of a logic, i.e., a logical language is a formal language, which has the following characteristics:

- The alphabet of a logical languages typically contains basic symbols that are used to indicate the basic (atomic) components of the (part of the) world the logic is supposed to describe. The alphabet is composed of two subsets: the logical symbols and the non logical symbols. Examples of such atomic objects are, individuals, functions, operators, truth-values, propositions, . . .

- The grammar of logical language define all the possible ways to construct complex phrases starting from simpler one. Logical grammar always define a grammar for building formulas which are phrases that denotes propositions, i.e., objects that can assume some truth value (as e.g., true, false, true in certain situation, true with probability of 3%, true/false in a period of time . . .) . Another important family of phrases which are usually defined in logic are terms which usually denotes objects of the world (e.g., cats, dogs, time points, quantities,

Alphabet

Alphabet

- The alphabet of a logical language is composed of two classes of symbols:
    - logical constants, whose formal interpretation is constant and fixed by the logic (e.g., $\wedge$, $\forall$, $=$, . . .
    - non logical symbols, whose formal interpretation is not fixed and can vary within a given range. They are not fixed by the logic and they must be defined by the "user".

Making an analogy with programming languages (say C, $C++$, $python$) logical constant are reserved words (their meaning is fixed by the interpreter/compiler); the non logical symbols are the identifiers that are introduced by the programmer for defining functions, variables, procedure, classes, attributes, methods, ... The meaning of these symbols is fixed by the programmer.

Alphabet - example

Logical Constant

- Propositional logic: $\wedge$, $\vee$, $\neg$, $\supset$, $\equiv$ and $\perp$, that stand for conjunction, disjunction, negation, implication, equivalence, and falsity. They are usually called propositional connectives

- Predicate logic (or first order logic): in addition to the propositional connectives we have $\forall$ and $\exists$, that stand for, "every object is such that ...", and "there is some object such that ...". They are usually called universal and existential quantifier

- Modal logic: in addition to the propositional connectives, we have $\square$ and $\lozenge$ that stand for "it is necessarily true that ..." and "it is possibly true that ...". They are usually called modal operators

Alphabet - example

Non logical symbols

- In propositional logic, non logical symbols are called propositional variables which represent ($=$ has intuitive interpretation) proposition. The proposition associated to each propositional variable is not fixed by the logic.
- In predicate logic. there are four families of non logical symbols
  - Variable symbols which represent any object
  - Constant symbols which represent specific objects
  - Function symbols which represent transformations on objects
  - Predicate symbols which represent relations between objects
- In modal logic, non logical symbols are the same as in propositional logic, i.e., propositional variables.

Intuitive interpretation of a logical language

Intuitive interpretation

While, non logical symbols do not have a fixed formal interpretation, they usually have a fixed intuitive interpretation. Consider for instance:

type	symbol	intuitive interpretation
propositional variable	rain	it's raining
constant symbol	$Moby\_Dick$	The whale of a novel by Melville
function symbol	$Color(x)$	the color of the object x
predicate symbol	$Friends(x, y)$	x and y are friends

Intuitive interpretation does not affect logic

The intuitive interpretation of the non logical symbols does not affect the logic itself. In other words, changing the intuitive interpretation does not affect the properties that will be proved in the logic. Similarly, replacing these logical symbols with less evocative symbols like r, m, $C(x)$ and $F(x, y)$ will not affect the logic.

Grammar - Example

Propositional logic

The grammar of propositional logic allow to define the unique class of phrases, called formulas (or well formed formulas), which denotes propositions.

```
FORMULA    :=    P  (P is a propositional variable)
           |  "("FORMULA "∧" FORMULA")"
           |  "("FORMULA "∨" FORMULA")"
           |  "("FORMULA "⊃" FORMULA")"
           |  "(" "¬" FORMULA ")"
```

Example (Well formed formulas)

$$P \wedge (Q \supset R) \qquad (P \supset (Q \supset R)) \vee P$$

The above formulas are well formed, because there is a sequence of application of grammar rules that allow to generate them. (Exercise: list the rules that allow to generate the above formulas).

Example (Non well formed formulas)

$$P(Q \supset R) \qquad (P \supset \vee P)$$

Grammar - Example

Language of first order logic

```
TERM      :=   x  (x is a variable symbol)
          |    c  (c is a constant symbol)
          |    f(TERM...TERM) (f is a function symbol)
FORMULA   :=   P(TERM...TERMINE) (P is a predicate symbol)
          |    FORMULA ∧ FORMULA
          |    FORMULA ∨ FORMULA
          |    FORMULA ⊃ FORMULA
          |    ¬ FORMULA
          |    ∀x FORMULA  (x is a variable symbol)
          |    ∃x FORMULA  (x is a variable symbol)
```

The rules defines two types of phrases, the terms and the formulas. Terms denote object (they are like noun phrases in natural language) while formulas denote propositions (they are like sentences in natural language).

Exercise

Give some examples of terms and formulas, and some examples of phrases which are neither terms nor formulas

Grammar - Example

The grammar for Description logics $\mathcal{ALC}$

```
FORMULA      :=    CONCEPT ⊑ CONCEPT
             |     CONCEPT ( INDIVIDUAL )
             |     ROLE ( INDIVIDUAL, INDIVIDUAL )
CONCEPT      :=    A  (A is a concept symbol)
             |     CONCEPT ⊔ CONCEPT
             |     CONCEPT ⊓ CONCEPT
             |     ¬ CONCEPT
             |     ∃ ROLE.CONCEPT
             |     ∀ ROLE.CONCEPT
ROLE         :=    R  (R is a role symbol)
INDIVIDUAL   :=    a  (a is a individual symbol)
```

Example (Concepts and formulas of DL)

CONCEPTS: $A \sqcap B$, $A \sqcup \exists R.C$, $\forall S(C \sqcup \forall R.D) \sqcap \neg A$

FORMULAS: $A \sqsubseteq B$, $A \sqsubseteq \exists R.B$, $A(a)$, $R(a, b)$, $\exists R.C(a)$,

Interpretation of complex formulas

Interpretation of complex formulas

The intuitive interpretation of complex formulas is done by combining the intuitive interpretation of the components of the formulas.

Example

The intuitive interpretation of the propositional formula

$$(raining \lor snowing) \supset \neg go\_to\_the\_beach$$

is obtained by composing the intuitive interpretations of the symbols that occurs in this formula. If the intuitive interpretation of the symbols are:

symbol	intuitive meaning
raining	it is raining
snowing	it is snowing
go_to_the_beach	we go to the beach
$\lor$	either it is the case of . . . or . . .
$\supset$	if . . . then . . .
$\neg$	it is not the case that . . .

then the above formula intuitively represent the proposition:

if (it is raining or it is snowing) then it is not the case that (we go to the beach)

Formal model

- **Class of models:** The models in which a logic is *formally interpreted* are the member of a class of algebraic structures each of which is an abstract representation of the relevant aspects of the (portion of the) world we want to formalize with this logic.
- **What do models represent?** Models represent only the components and aspects of the worlds with are relevant to a certain analysis, and abstract away from irrelecant facts. For instance, if we are interested in the everage temperature of each day, we can represent time with the natural numbers and a function that associates to each natural number a floating point number (the average temperature of the day corresponding to the point)
- **Applicability of a model** Since the (real) world is quite complex, in the construction of the formal model, we usually make a number of simplifying assumptions that bound the usability of the logic to the cases in which these assumptions are verified. For instance if we take integers as a formal model of time, then we cannot use this model to represent continuos change.
- **What represents a single model** Each element of the class represent a single possible (or impossible) state of the world. The class of models of a logic will represent all the (im)possible states of the world.

Formal interpretation

- Given a structure S, the formal interpretation of a logical language in S is a function that associates an element of S to any non logical symbol of the alphabet.

- the formal interpretation on the algebraic structure is the parallel (or better is the formalization) of the intuitive interpretation in the real world.

- The formal interpretation is specified only for the non logical symbols, while the formal interpretation of the logical symbols is fixed by the logic.

- The formal interpretation of complex expressions e obtained by a combination of the sub-expressions $e_1, \ldots e_n)$ is unequivocally determined as a function of the formal interpretations of the sub-components $e_1, \ldots, e_n$.

Truth in a model (= satisfiable in a mode)

As clarified at the beginning, the goal of logic is the formalization of what is true/false on a particular world. The main objective of the formal interpretation is that it allows to define when a formula is true in model (model = structure + interpretation).
Every logic therefore define the satisfiability relation, denoted by $\models$ between models and formulas
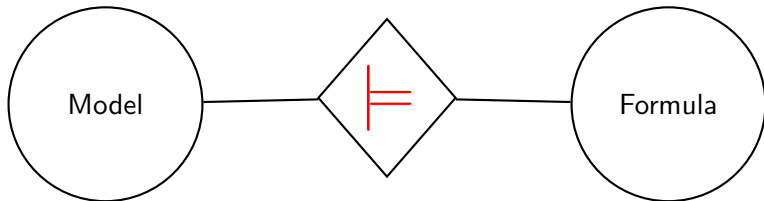
If M is a model and ϕ a formula, tehn

$$M \models \phi$$

stands for the fact that M satisfies ϕ, or equivalently that ϕ is true in M

(Un)-Satisfiability and Validity

On the basis of truth in a model ($\models$) the following concepts are defined in any logic

- ϕ is satisfiable if there is a model in which it is true i.e. if there is an M such that $M \models \phi$
- ϕ is un-satisfiable if it is not satisfiable, i.e., there are no models that makes ϕ true
- ϕ is valid if is true in all the models.

Logical consequence

The notion of logical consequence is defined on the basis of the notion of truth in a model. Intuitively, a formula ϕ is a logical consequence of a set of formulas Γ (sometime called assumptions) if such a formula is true under this set of assumptions.
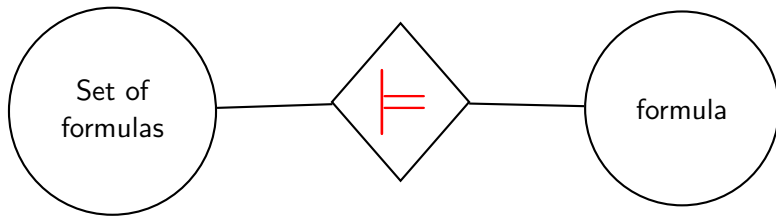
Notice that the two relations, "truth in a model" and "logical consequence" are denoted by the same symbol (this should remind you that they are tightly connected). Formally $\Gamma \models \phi$ holds when

$$\text{For all } M, \text{ if } M \models \Gamma \text{ then } M \models \phi$$

In words: ϕ is true in all the possible situations in which all the formulas in Γ are true.

Checking logical consequence

Problem

Does exist an algorithm that checks if a formula ϕ is a logical consequence of a set of formulas Γ?

Solution (1)

If Γ is finite and the set of models of the logic is finite, then it is possible to directly apply the definition by checking for every model M, that if $M \models \Gamma$ then, $M \models \phi$.

Solution (2)

When Γ is infinite or the set of models is infinite, then solution (1) is not applicable as it would run infinitely. An alternative solution could be to generate, starting from Γ, all its logical consequences by applying a set of rules.

Checking logical consequence

Propositional logic The method based on truth tables can be used to check logical consequence by enumerating all the models of Γ and ϕ and checking if every time all the formulas in Γ are true then ϕ is also true. This is possible because, when Γ is finite then there are a finite number of models.

First order logic A first order language in general has an infinite number of interpretations. Therefore, to check logical consequence, it is not possible to apply a method that enumerates all the possible models, as in truth tables.

Modal logic present the same problem of first order logic. In general for a set of formulas Γ there are infinite number of models, which implies that a method that enumerates all the models is not effective.

An alternative method for determining if a formula is a logical consequence of a set of formulas is based on inference rules. An inference rule is a rewriting rules that takes a set of formulas and transform it in another formulas. The following are examples of inference rules.

$$\frac{\phi \quad \psi}{\phi \land \psi} \qquad \frac{\phi \quad \psi}{\phi \supset \psi} \qquad \frac{\forall x.\phi(x)}{\phi(c)} \qquad \frac{\exists x.\phi(x)}{\phi(d)}$$

Differently from truth table, which apply a brute force exhaustive analysis not interpretable by humans, the deductive method simulates human argumentation and provides also an understandable explanation (i.e., a deduction) of the reason why a formula is a logical consequence of a set of formulas.

Checking logical consequence
Deductive methods - Example

Example

Let $\Gamma = \{r, (r \vee s) \supset \neg b\}$. The following is a deduction (an explanation of) the fact that $\neg b$ is a logical consequence of Γ, i.e., that $\Gamma \models \neg b$ that uses the following inference rules:

$$\frac{\phi \quad \phi \supset \psi}{\psi}(MP) \quad \frac{\phi}{\phi \vee \psi}(OR - intro)$$

(1)	r	Belongs to Γ
(2)	$r \vee s$	by applying (OR-intro) to (1)
(3)	$(r \vee s) \supset \neg b$	Belongs to Γ
(5)	$\neg b$	By applying (MP) to (2) and (3)

Introduction to first order logic for knowledge representation

Part 2 - Introduction to Propositional Logic

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Propositional logic - Intuition

- Propositional logic is the logic of propositions
- a proposition can be true or false in the state of the world.
- the same proposition can be expressed in different ways. E.g. "B. Obama is drinking a bier", " The U.S.A. president is drinking a bier" and "B. Obama si sta facendo una birra" express the same proposition.
- The language of propositional logic allows us to express propositions.

Propositional logic language

Definition (Propositional alphabet)

Logical symbols $\neg$, $\wedge$, $\vee$ $\supset$, and $\equiv$

Non logical symbols A set $\mathcal{P}$ of symbols called propositional variables

Separator symbols "(" and ")"

Definition (Well formed formulas or simply formulas)

- every $P \in \mathcal{P}$ is an atomic formula
- every atomic formula is a formula
- if A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$ $A \supset B$, and $A \equiv B$ are formulas

Formulas cont'd

Example ((non) formulas)

Formulas	Non formulas
$P \supset Q$	PQ
$P \supset (Q \supset R)$	$(P \supset \wedge ((Q \supset R)$
$P \wedge Q \supset R$	$P \wedge Q \supset \neg R \neg$

Problem

The formula $P \wedge Q \supset R$ can be read in two ways:

1. $(P \wedge Q) \supset R$
2. $P \wedge (Q \supset R)$

Symbol priority

$\neg$ has higher priority, then $\wedge$, $\vee$, $\supset$ and $\equiv$. Parenthesis can be used around formulas to stress or change the priority.

Formulas as trees

Tree form of a formula

A formula can be seen as a tree. Leaf nodes are associated to propositional variables, while intermediate (non-leaf) nodes are associated to connectives.

For instance the formula $(A \wedge \neg B) \equiv (B \supset C)$ can be represented as the tree
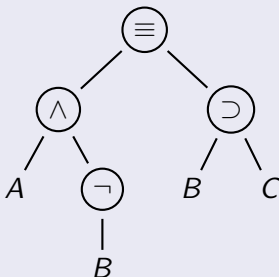
Subformulas

Definition

(Proper) Subformula

- A is a subformula of itself
- A and B are subformulas of $A \wedge B$, $A \vee B$ $A \supset B$, e $A \equiv B$
- A is a subformula of $\neg A$
- if A is a subformula of B and B is a subformula of C, then A is a subformula of C.
- A is a proper subformula of B if A is a subformula of B and A is different from B.

Remark

The subformulas of a formula represented as a tree correspond to all the different subtrees of the tree associated to the formula, one for each node.

Subformulas

Example

The subformulas of $(p \supset (q \lor r)) \supset (p \land \neg p)$ are

$(p \supset (q \lor r)) \supset (p \land \neg p)$
$(p \supset (q \lor r))$
$p \land \neg p$
p
$\neg p$
$q \lor r$
q
r
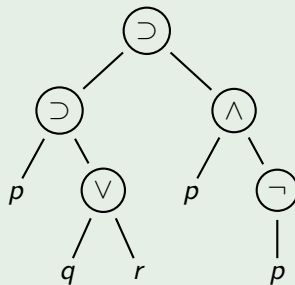
Proposition

Every formula has a finite number of subformulas

Definition (Interpretation)

A Propositional interpretation is a function $\mathcal{I} : \mathcal{P} \rightarrow \{\text{True}, \text{False}\}$

Remark

If $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$, then there are $2^{|\mathcal{P}|}$ different interpretations, i.e. all the different subsets of $\mathcal{P}$. If $|\mathcal{P}|$ is finite then there is a finite number of interpretations.

Remark

A propositional interpretation can be thought as a subset S of $\mathcal{P}$, and $\mathcal{I}$ is the characteristic function of S, i.e., $A \in S$ iff $\mathcal{I}(A) = \text{True}$.

Interpretation of Propositional Logic

Example

	p	q	r	Set theoretic representation
$\mathcal{I}_1$	True	True	True	$\{p, q, r\}$
$\mathcal{I}_2$	True	True	False	$\{p, q\}$
$\mathcal{I}_3$	True	False	True	$\{p, r\}$
$\mathcal{I}_4$	True	False	False	$\{p\}$
$\mathcal{I}_5$	False	True	True	$\{q, r\}$
$\mathcal{I}_6$	False	True	False	$\{q\}$
$\mathcal{I}_7$	False	False	True	$\{r\}$
$\mathcal{I}_8$	False	False	False	$\{\}$

Satisfiability of a propositional formula

Definition ($\mathcal{I}$ satisfies a formula, $\mathcal{I} \models A$)

A formula A is true in/satisfied by an interpretation $\mathcal{I}$, in symbols $\mathcal{I} \models A$, according to the following inductive definition:

- If $P \in \mathcal{P}$, $\mathcal{I} \models P$ if $\mathcal{I}(P) =$ True.
- $\mathcal{I} \models \neg A$ if not $\mathcal{I} \models A$ (also written $\mathcal{I} \not\models A$
- $\mathcal{I} \models A \wedge B$ if, $\mathcal{I} \models A$ and $\mathcal{I} \models B$
- $\mathcal{I} \models A \vee B$ if, $\mathcal{I} \models A$ or $\mathcal{I} \models B$
- $\mathcal{I} \models A \supset B$ if, when $\mathcal{I} \models A$ then $\mathcal{I} \models B$
- $\mathcal{I} \models A \equiv B$ if, $\mathcal{I} \models A$ iff $\mathcal{I} \models B$

Satisfiability of a propositional formula

Example (interpretation)

$\mathcal{P} = \{P, Q\}$, $I(P) = \textit{True}$ and $\mathcal{I}(Q) = \textit{False}$ can be also expressed with $I = \{P\}$.

Example (Satisfiability)

Let $I = \{P\}$. Check if $\mathcal{I} \models (P \wedge Q) \vee (R \supset S)$:
Replace each occurrence of each primitive propositions of the formula with the truth value assigned by $\mathcal{I}$, and apply the definition for connectives.

$$(\text{True} \wedge \text{False}) \vee (\textit{False} \supset \text{False}) \tag{1}$$

$$\text{False} \vee \text{True} \tag{2}$$

$$\text{True} \tag{3}$$

Proposition

If for any propositional variable P appearing in a formula A,
$\mathcal{I}(P) = \mathcal{I}'(P)$, *then* $\mathcal{I} \models A$ *iff* $\mathcal{I}' \models A$

Definition

A formula A is

Valid if for all interpretations $\mathcal{I}$, $\mathcal{I} \models A$

Satisfiable if there is an interpretations $\mathcal{I}$ s.t., $\mathcal{I} \models A$

Unsatisfiable if for no interpretations $\mathcal{I}$, $\mathcal{I} \models A$

Proposition

A Valid $\longrightarrow$ A satisfiable $\longleftrightarrow$ A not unsatisfiable

A unsatisfiable $\longleftrightarrow$ A not satisfiable $\longrightarrow$ A not Valid

Valid, Satisfiable, and Unsatisfiable formulas - II

Proposition

if A is	then ¬A is
Valid	Unsatisfiable
Satisfiable	not Valid
not Valid	Satisfiable
Unsatisfiable	Valid

Chesking Validity and (un)satisfiability of a formula

Truth Table

Checking (un)satisfiability and validity of a formula A can be done by enumerating all the interpretations which are relevant for S, and for each interpretation I check if $I \models A$.

Example (of truth table)

A	B	C	$A \supset (B \vee \neg C)$
true	true	true	true
true	true	false	true
true	false	true	false
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

Valid, Satisfiable, and Unsatisfiable formulas

Example (Excercise)

$$\text{Satisfiable} \begin{cases} \left. \begin{array}{l} A \supset A \\ A \lor \neg A \\ \neg\neg A \equiv A \\ \neg(A \land \neg A) \\ A \land B \supset A \\ A \supset A \lor B \end{array} \right\} \text{Valid} \\ \left. \begin{array}{l} A \lor B \\ A \supset B \\ \neg(A \lor B) \supset C \end{array} \right\} \\ \left. \begin{array}{l} A \land \neg A \\ \neg(A \supset A) \\ A \equiv \neg A \\ \neg(A \equiv A) \end{array} \right\} \text{Non Valid} \end{cases}$$

Uatisfiable

Prove that the blue formulas are valid, that the magenta formulas are satisfiable but not valid, and that the red formulas are unsatisfiable.

Valid, Satisfiable, and Unsatisfiable sets of formulas

Definition

A set of formulas Γ is

Valid if for all interpretations $\mathcal{I}$, $\mathcal{I} \models A$ for all formulas $A \in \Gamma$

Satisfiable if there is an interpretations $\mathcal{I}$, $\mathcal{I} \models A$ for all $A \in \Gamma$

Unsatisfiable if for no interpretations $\mathcal{I}$,, s.t. $\mathcal{I} \models A$ for all $A \in \Gamma$
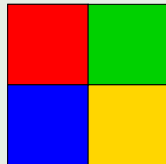
Proposition

For any finite set of formulas Γ, (i.e., $\Gamma = \{A_1, \ldots, A_n\}$ for some $n \geq 1$), Γ is valid (resp. satisfiable and unsatisfiable) if and only if $A_1 \wedge \cdots \wedge A_n$ is valid (resp, satisfiable and unsatisfiable).

Formalization in Propositional Logic

Example (The colored blanket)

- $\mathcal{P} = \{B, R, Y, G\}$
- the intuitive interpretation of B (R, Y, and G) is that the blanket is completely blue (red, yellow and green)

Exercise

Find all the interpretations that, according to the intuitive interpretation given above, represent a possible situation. Consider the two cases in which

1. the blanket is either completely yellow, red, blue or green (i.e, yellow, red, blue or green are the only allowed colors);

2. the blanket is composed of an arbitrary of colors;

Formalization in Propositional Logic

Exercise (Solution)

- $\mathcal{I}_1 = \{B\}$ corrisponding to ;
- $\mathcal{I}_2 = \{Y\}$ corrisponding to ;
- $\mathcal{I}_3 = \{R\}$ corrisponding to ;
- $\mathcal{I}_4 = \{G\}$ corrisponding to ;
- $\mathcal{I}_5 = \emptyset$ corrisponding to any blanket that is not monochrome, e.g. , ...
- $\mathcal{I}_6 = \{R, B\}$ does not correspond to any blanket, since a blanket cannot be both completely blue and red. More in general all the interpretations that satisfies more than one proposition do not correspond to any real situation.

Logical consequence

Definition (Logical consequence)

A formula A is a logical consequence of a set of formulas Γ, in symbols

$$\Gamma \models A$$

Iff for any interpretation $\mathcal{I}$ that satisfies all the formulas in Γ, $\mathcal{I}$ satisfies A,

Example (Logical consequence)

- $p \models p \vee q$
- $q \vee p \models p \vee q$
- $p \vee q, p \supset r, q \supset r \models r$
- $p \supset q, p \models q$
- $p, \neg p \models q$

Logical consequence

Example

Proof of $p \models p \vee q$ Suppose that $\mathcal{I} \models p$, then by definition $\mathcal{I} \models p \vee q$.

Proof of $q \vee p \models p \vee q$ Suppose that $\mathcal{I} \models q \vee p$, then either $\mathcal{I} \models q$ or $\mathcal{I} \models p$. In both cases we have that $\mathcal{I} \models p \vee q$.

Proof of $p \vee q, p \supset r, q \supset r \models r$ Suppose that $\mathcal{I} \models p \vee q$ and $\mathcal{I} \models p \supset r$ and $\mathcal{I} \models q \supset r$. Then either $\mathcal{I} \models p$ or $\mathcal{I} \models q$. In the first case, since $\mathcal{I} \models p \supset r$, then $\mathcal{I} \models r$, In the second case, since $\mathcal{I} \models q \supset r$, then $\mathcal{I} \models r$.

Proof of $p, \neg p \models q$ Suppose that $\mathcal{I} \models \neg p$, then not $\mathcal{I} \models p$, which implies that there is no $\mathcal{I}$ such that $\mathcal{I} \models p$ and $\mathcal{I} \models \neg p$. This implies that all the interpretations that satisfy p and $\neg p$ (actually none) satisfy also p.

Proof of $(p \wedge q) \vee (\neg p \wedge \neg q) \models p \equiv q)$ Left by exercise

Proof of $(p \supset q) \models \neg p \vee q$ Left by exercise

Introduction to first order logic for knowledge representation

Part 3 - Introduction to First order logic

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Outline

- Why First Order Logic (FOL)?
- Syntax and Semantics of FOL;
- First Order Theories;
- "Fun" with sentences...

Expressivity of propositional logic - I

Question

Try to express in Propositional Logic the following statements:

- Mary is a person
- John is a person
- Mary is mortal
- Mary and John are siblings

Expressivity of propositional logic - I

Question

Try to express in Propositional Logic the following statements:

- Mary is a person
- John is a person
- Mary is mortal
- Mary and John are siblings

A solution

Through atomic propositions:

- `Mary-is-a-person`
- `John-is-a-person`
- `Mary-is-mortal`
- `Mary-and-John-are-siblings`

Problem with previous solution

- `Mary-is-a-person`
- `John-is-a-person`
- `Mary-is-mortal`
- `Mary-and-John-are-siblings`

Problem with previous solution

- `Mary-is-a-person`
- `John-is-a-person`
- `Mary-is-mortal`
- `Mary-and-John-are-siblings`

How do we link `Mary` of the first sentence to `Mary` of the third sentence? Same with `John`. How do we link `Mary` and `Mary-and-John`?

Expressivity of propositional logic - II

Question

Try to express in Propositional Logic the following statements:

- All persons are mortal;
- There is a person who is a spy.

Expressivity of propositional logic - II

Question

Try to express in Propositional Logic the following statements:

- All persons are mortal;
- There is a person who is a spy.

A solution

We can give all people a name and express this fact through atomic propositions:

- Mary-is-mortal ∧ John-is-mortal ∧ Chris-is-mortal
  ∧...∧ Michael-is-mortal
- Mary-is-a-spy ∨John-is-a-spy ∨Chris-is-a-spy
  ∨...∨ Michael-is-a-spy

Problem with previous solution

- Mary-is-mortal ∧ John-is-mortal ∧ Chris-is-mortal
  ∧...∧ Michael-is-mortal
- Mary-is-a-spy ∨John-is-a-spy ∨Chris-is-a-spy
  ∨...∨ Michael-is-a-spy

Problem with previous solution

- Mary-is-mortal $\wedge$ John-is-mortal $\wedge$ Chris-is-mortal $\wedge \ldots \wedge$ Michael-is-mortal
- Mary-is-a-spy $\vee$ John-is-a-spy $\vee$ Chris-is-a-spy $\vee \ldots \vee$ Michael-is-a-spy

The representation is not compact and generalization patterns are difficult to express.

- Mary-is-mortal $\wedge$ John-is-mortal $\wedge$ Chris-is-mortal
  $\wedge \ldots \wedge$ Michael-is-mortal
- Mary-is-a-spy $\vee$ John-is-a-spy $\vee$ Chris-is-a-spy
  $\vee \ldots \vee$ Michael-is-a-spy

The representation is not compact and generalization patterns are difficult to express.

What is we do not know all the people in our "universe"? How can we express the statement independently from the people in the "universe"?

Question

Try to express in Propositional Logic the following statements:

- Every natural number is either even or odd

Expressivity of propositional logic - III

Question

Try to express in Propositional Logic the following statements:

- Every natural number is either even or odd

A solution

We can use two families of propositions $even_i$ and odd_i for every $i \geq 1$, and use the set of formulas

$$\{odd_i \vee even_i | i \geq 1\}$$

$$\{odd_i \lor even_i | i \geq 1\}$$

What happens if we want to state this in one single formula? To do this we would need to write an infinite formula like:

$$(odd_1 \lor even_1) \land (odd_2 \lor even_2) \land \ldots$$

and this cannot be done in propositional logic.

Expressivity of propositional logic -IV

Question

Express the statements:

- the father of Luca is Italian

Solution (Partial)

- `mario-is-father-of-luca` ⊃ `mario-is-italian`
- `michele-is-father-of-luca` ⊃ `michele-is-italian`
- ...

Problem with previous solution

- `mario-is-father-of-luca` ⊃ `mario-is-italian`
- `michele-is-father-of-luca` ⊃ `michele-is-italian`
- ...

This statement strictly depend from a fixed set of people. What happens if we want to make this statement independently of the set of persons we have in our universe?

Why first order logic?

Because it provides a way of representing information like the following one:

1. Mary is a person;
2. John is a person;
3. Mary is mortal;
4. Mary and John are siblings
5. Every person is mortal;
6. There is a person who is a spy;
7. Every natural number is either even or odd;
8. The father of Luca is Italian

Why first order logic?

Because it provides a way of representing information like the following one:

1. Mary is a person;
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6. There is a person who is a spy;
7. Every natural number is either even or odd;
8. The father of Luca is Italian

and also to infer the third one from the first one and the fifth one.

First order logic

Whereas propositional logic assumes world contains facts,
first-order logic (like natural language) assumes the world contains:

- Constants: mary, john, 1, 2, 3, red, blue, world war 1, world war 2, 18th Century...
- Predicates: Mortal, Round, Prime, Brother of, Bigger than, Inside, Part of, Has color, Occurred after, Owns, Comes between, ...
- Functions: Father of, Best friend, Third inning of, One more than, End of, ...

Constants and Predicates

- Mary is a person
- John is a person
- Mary is mortal
- Mary and John are siblings

In FOL it is possible to build an atomic propositions by applying a predicate to constants

- *Person*(*mary*)
- *Person*(*john*)
- *Mortal*(*mary*)
- *Siblings*(*mary*, *john*)

Quantifiers and variables

- Every person is mortal;
- There is a person who is a spy;
- Every natural number is either even or odd;

In FOL it is possible to build propositions by applying universal (existential) quantifiers to variables. This allows to quantify to arbitrary objects of the universe.

- $\forall x. Person(x) \supset Mortal(x)$;
- $\exists x. Person(x) \supset Spy(x)$;
- $\forall x. (Odd(x) \lor Even(x))$

- The father of Luca is Italian.

In FOL it is possible to build propositions by applying a function to a constant, and then a predicate to the resulting object.

- *Italian(fatherOf(Mario))*

Syntax of FOL

Logical symbols

- the logical constant $\perp$
- propositional logical connectives $\wedge$, $\vee$, $\supset$, $\neg$, $\equiv$
- the quantifiers $\forall$, $\exists$
- an infinite set of variable symbols $x_1, x_2, \ldots$
- the equality symbol $=$. (optional)

Non Logical symbols

- a set $c_1, c_2, \ldots$ of constant symbols
- a set $f_1, f_2, \ldots$ of functional symbols each of which is associated with its *arity* (i.e., number of arguments)
- a set $P_1, P_2, \ldots$ of *relational symbols* each of which is associated with its *arity* (i.e., number of arguments)

Terms and formulas of FOL

Terms

- every constant c_i and every variable x_i is a term;
- if $t_1, \ldots, t_n$ are terms and f_i is a functional symbol of arity equal to n, then $f(t_1, \ldots, t_n)$ is a term

Well formed formulas

- if t_1 and t_2 are terms then $t_1 = t_2$ is a formula
- If $t_1, \ldots, t_n$ are terms and P_i is relational symbol of arity equal to n, then $P_i(t_1, \ldots, t_n)$ is formula
- if A and B are formulas then $\bot$, $A \wedge B$, $A \supset B$, $A \vee B$ $\neg A$ are formulas
- if A is a formula and x a variable, then $\forall x.A$ and $\exists x.A$ are formulas.

Examples of terms and formulas

Example (Terms)

- x_i,
- c_i,
- $f_i(x_j, c_k)$, and
- $f(g(x, y), h(x, y, z), y)$

Example (formulas)

- $f(a, b) = c$,
- $P(c_1)$,
- $\exists x(A(x) \lor B(y))$, and
- $P(x) \supset \exists y. Q(x, y)$.

An example of representation in FOL

Example (Language)

constants	functions (arity)	Predicate (arity)
Aldo	mark (2)	attend (2)
Bruno	best-friend (1)	friend (2)
Carlo		student (1)
MathLogic		course (1)
DataBase		less-than (2)
0, 1, . . . , 10		

Example (Terms)

Intuitive meaning

an individual named Aldo
the mark 1
Bruno's best friend
anything
Bruno's mark in MathLogic
somebody's mark in DataBase
Bruno's best friend mark in MathLogic

An example of representation in FOL

Example (Language)

constants	functions (arity)	Predicate (arity)
Aldo	mark (2)	attend (2)
Bruno	best-friend (1)	friend (2)
Carlo		student (1)
MathLogic		course (1)
DataBase		less-than (2)
0, 1, . . . , 10		

Example (Terms)

Intuitive meaning	term
an individual named Aldo	Aldo
the mark 1	1
Bruno's best friend	best-friend(Bruno)
anything	x
Bruno's mark in MathLogic	mark(Bruno,MathLogic)
somebody's mark in DataBase	mark(x,DataBase)
Bruno's best friend mark in MathLogic	mark(best-friend(Bruno),MathLogic)

Example (Formulas)

Intuitive meaning

Aldo and Bruno are the same person
Carlo is a person and MathLogic is a course
Aldo attends MathLogic
Courses are attended only by students
every course is attended by somebody
every student attends something
a student who attends all the courses
every course has at least two attenders
Aldo's best friend attend the same courses
  attended by Aldo
best-friend is symmetric
Aldo and his best friend have the same mark
  in MathLogic
A student can attend at most two courses

An example of representation in FOL (cont'd)

Intuitive meaning	Formula
Aldo and Bruno are the same person	$Aldo = Bruno$
Carlo is a person and MathLogic is a course	$person(Carlo) \land course(MathLogic)$
Aldo attends MathLogic	$attend(Aldo, MathLogic)$
Courses are attended only by students	$\forall x(attend(x, y) \supset course(y) \supset student(x))$
every course is attended by somebody	$\forall x(course(x) \supset \exists y\ attend(y, x))$
every student attends something	$\forall x(student(x) \supset \exists y\ attend(x, y))$
a student who attends all the courses	$\exists x(student(x) \land \forall y(course(y) \supset attend(x, y)))$
every course has at least two attenders	$\forall x(course(x) \supset \exists y \exists z(attend(y, x) \land attend(z, x) \land \neg y = z))$
Aldo's best friend attend the same courses attended by Aldo	$\forall x(attend(Aldo, x) \supset attend(best\text{-}friend(Aldo), x))$
best-friend is symmetric	$\forall x(best\text{-}friend(best\text{-}friend(x)) = x)$
Aldo and his best friend have the same mark in MathLogic	$mark(best\text{-}friend(Aldo), MathLogic) = mark(Aldo, MathLogic)$
A student can attend at most two courses	$\forall x \forall y \forall z \forall w(attend(x, y) \land attend(x, z) \land attend(x, w) \supset (y = z \lor z = w \lor y = w))$

Common Mistakes

- Use of $\wedge$ with $\forall$

  $\forall x\ (WorksAt(FBK, x) \wedge Smart(x))$

Common Mistakes

- Use of $\wedge$ with $\forall$

  $\forall x \, (WorksAt(FBK, x) \wedge Smart(x))$ means "Everyone works at FBK and everyone is smart"

Common Mistakes

- Use of $\land$ with $\forall$

  $\forall x \ (WorksAt(FBK, x) \land Smart(x))$ means "Everyone works at FBK and everyone is smart"

  "Everyone working at FBK is smart" is formalized as
  $\forall x \ (WorksAt(FBK, x) \supset Smart(x))$

Common Mistakes

- Use of ∧ with ∀

  $\forall x \, (WorksAt(FBK, x) \land Smart(x))$ means "Everyone works at FBK and everyone is smart"

  "Everyone working at FBK is smart" is formalized as
  $\forall x \, (WorksAt(FBK, x) \supset Smart(x))$

- Use of ⊃ with ∃

  $\exists x \, (WorksAt(FBK, x) \supset Smart(x))$

Common Mistakes

- Use of ∧ with ∀

  $\forall x \, (WorksAt(FBK, x) \wedge Smart(x))$ means "Everyone works at FBK and everyone is smart"

  "Everyone working at FBK is smart" is formalized as
  $\forall x \, (WorksAt(FBK, x) \supset Smart(x))$

- Use of ⊃ with ∃

  $\exists x \, (WorksAt(FBK, x) \supset Smart(x))$ mans "There is a person so that if (s)he works at FBK then (s)he is smart" and this is true as soon as there is at last an x who does not work at FBK

Common Mistakes

- Use of $\wedge$ with $\forall$

  $\forall x \, (WorksAt(FBK, x) \wedge Smart(x))$ means "Everyone works at FBK and everyone is smart"

  "Everyone working at FBK is smart" is formalized as
  $\forall x \, (WorksAt(FBK, x) \supset Smart(x))$

- Use of $\supset$ with $\exists$

  $\exists x \, (WorksAt(FBK, x) \supset Smart(x))$ mans "There is a person so that if (s)he works at FBK then (s)he is smart" and this is true as soon as there is at last an x who does not work at FBK

  "There is an FBK-working smart person" is formalized as
  $\exists x \, (WorksAt(FBK, x) \wedge Smart(x))$

Example

Represent the statement at most 2 students attend the KR course

$$\forall x_1 \forall x_2 \forall x_3 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge attend(x_2, KR) \supset$$
$$x_1 = x_2 \vee x_2 = x_3 \vee x_1 = x_3)$$

At most n ...

$$\forall x_1 \ldots x_{n+1} \left(\bigwedge_{i=1}^{n+1} \phi(x_i) \supset \bigvee_{i \neq j=1}^{n+1} x_i = x_j \right)$$

Representing variations quantifiers in FOL

Example

Represent the statement at least 2 students attend the KR course

$$\exists x_1 \exists x_2 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge x_1 \neq x_2)$$

At least n . . .

$$\exists x_1 \ldots x_n \left(\bigwedge_{i=1}^{n} \phi(x_i) \wedge \bigwedge_{i \neq j=1}^{n} x_i \neq x_j \right)$$

Semantics of FOL

FOL interpretation for a language L

A first order interpretation for the language
$L = \langle c_1, c_2, \ldots, f_1, f_2, \ldots, P_1, P_2, \ldots \rangle$ is a pair $\langle \Delta, \mathcal{I} \rangle$ where

- Δ is a non empty set called interpretation domain
- $\mathcal{I}$ is is a function, called interpretation function
  - $\mathcal{I}(c_i) \in \Delta$ (elements of the domain)
  - $\mathcal{I}(f_i) : \Delta^n \to \Delta$ (n-ary function on the domain)
  - $\mathcal{I}(P_i) \subseteq \Delta^n$ (n-ary relation on the domain)

where n is the arity of f_i and P_i.

Example of interpretation

Example (Of interpretation)

Symbols

Constants: *alice*, *bob*, *carol*, *robert*

Function: *mother-of* (with arity equal to 1)

Predicate: *friends* (with arity equal to 2)
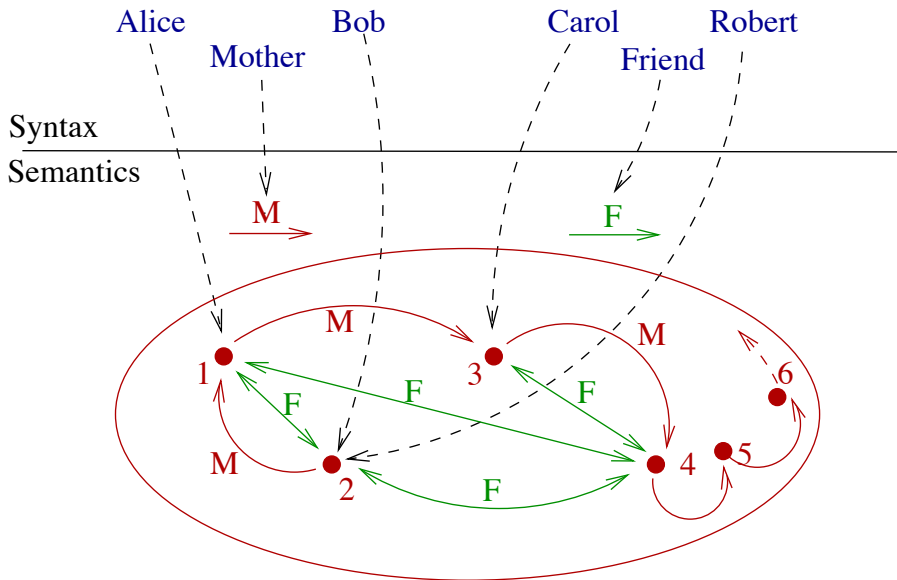

Domain

$\Delta = \{1, 2, 3, 4, \dots\}$

Interpretation

$\mathcal{I}(alice) = 1$, $\mathcal{I}(bob) = 2$, $\mathcal{I}(carol) = 3$,

$\mathcal{I}(robert) = 2$

$\mathcal{I}(mother\text{-}of) = M$
$$\begin{aligned} M(1) &= 3 \\ M(2) &= 1 \\ M(3) &= 4 \\ M(n) &= n+1 \text{ for } n \geq 4 \end{aligned}$$

$\mathcal{I}(friends) = F = \left\{ \begin{array}{lll} \langle 1, 2 \rangle, & \langle 2, 1 \rangle, & \langle 3, 4 \rangle, \\ \langle 4, 3 \rangle, & \langle 4, 2 \rangle, & \langle 2, 4 \rangle, \\ \langle 4, 1 \rangle, & \langle 1, 4 \rangle, & \langle 4, 4 \rangle \end{array} \right\}$

Example (cont'd)

Interpretation of terms

Definition (Assignment)

An assignment a is a function from the set of variables to Δ.

$a[x/d]$ denotes the assignment that coincides with a on all the variables but x, which is associated to d.

Definition (Interpretation of terms)

The interpretation of a term t w.r.t. the assignment a, in symbols $\mathcal{I}(t)[a]$ is recursively defined as follows:

$$
\begin{aligned}
\mathcal{I}(x_i)[a] &= a(x_i) \\
\mathcal{I}(c_i)[a] &= \mathcal{I}(c_i) \\
\mathcal{I}(f(t_1, \ldots, t_n))[a] &= \mathcal{I}(f)(\mathcal{I}(t_1)[a], \ldots, \mathcal{I}(t_n)[a])
\end{aligned}
$$

FOL Satisfiability of formulas

Definition (Satisfiability of a formula w.r.t. an assignment)

An interpretation $\mathcal{I}$ satisfies a formula ϕ w.r.t. the assignment a according to the following rules:

$$
\begin{aligned}
\mathcal{I} \models t_1 = t_2[a] \quad &\text{iff} \quad \mathcal{I}(t_1)[a] = \mathcal{I}(t_2)[a] \\
\mathcal{I} \models P(t_1, \ldots, t_n)[a] \quad &\text{iff} \quad \langle \mathcal{I}(t_1)[a], \ldots, \mathcal{I}(t_n)[a] \rangle \in \mathcal{I}(P) \\
\mathcal{I} \models \phi \wedge \psi[a] \quad &\text{iff} \quad \mathcal{I} \models \phi[a] \text{ and } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \vee \psi[a] \quad &\text{iff} \quad \mathcal{I} \models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \supset \psi[a] \quad &\text{iff} \quad \mathcal{I} \not\models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \neg\phi[a] \quad &\text{iff} \quad \mathcal{I} \not\models \phi[a] \\
\mathcal{I} \models \phi \equiv \psi[a] \quad &\text{iff} \quad \mathcal{I} \models \phi[a] \text{ iff } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \exists x \phi[a] \quad &\text{iff} \quad \text{there is a } d \in \Delta \text{ such that } \mathcal{I} \models \phi[a[x/d]] \\
\mathcal{I} \models \forall x \phi[a] \quad &\text{iff} \quad \text{for all } d \in \Delta, \mathcal{I} \models \phi[a[x/d]]
\end{aligned}
$$

Example (cont'd)

Exercise

Check the satisfiability of the following statements, considering the interpretation defined few slides ago:

1. $\mathcal{I} \models Alice = Bob[a]$
2. $\mathcal{I} \models Robert = Bob[a]$
3. $\mathcal{I} \models x = Bob[a[x/2]$

Example (cont.)

$\mathcal{I}(\text{mother-of}(\text{alice}))[a] = 3$

$\mathcal{I}(\text{mother-of}(x))[a[x/4]] = 5$

$\mathcal{I}(\text{friends}(x, y)) =$

$x :=$	$y :=$
1	2
2	1
4	1
1	4
4	2
2	4
4	3
3	4
4	4

$\mathcal{I}(\text{friends}(x, x)) =$

$x :=$
4

$\mathcal{I}(\text{friends}(x, y) \land x = y) =$

$x :=$	$y :=$
4	4

$\mathcal{I}(\exists x\, \text{friends}(x, y)) =$

$y :=$
2
1
4
3

$\mathcal{I}(\forall x\, \text{friends}(x, y)) =$

$y :=$
4

Free variable and free terms

Intuition

A free occurrence of a variable x is an occurrence of x which is not bounded by a (universal or existential) quantifier.

Definition (Free occurrence)

- any occurrence of x in t_k is free in $P(t_1, \ldots, t_k, \ldots, t_n)$
- any free occurrence of x in ϕ or in ψ is also fee in $\phi \wedge \psi$, $\psi \vee \phi$, $\psi \supset \phi$, and $\neg \phi$
- any free occurrence of x in ϕ, is free in $\forall y.\phi$ and $\exists y.\phi$ if y is distinct from x.

Definition (Ground/Closed Formula)

A formula ϕ is ground if it does not contain any variable. A formula is closed if it does not contain free occurrences of variables.

Free variable and free terms

A variable x is free in ϕ (denote by $\phi(x)$) if there is at least a free occurrence of x in ϕ.

Free variables represents individuals which must be instantiated to make the formula a meaningful proposition.

- x is free in $friends(alice, x)$.
- x is free in $P(x) \supset \forall x. Q(x)$ (the occurrence of x in red is free the one in green is not free.

Definition (Term free for a variable)

A term is free for x in ϕ, if all the occurrences of x in ϕ are not in the scope of a quantifier for a variable occurring in t.

An occurrence of a variable x can be safely instantiated by a term free for x in a formula ϕ,

If you replace x with a terms which is not free for x in ϕ, you can have unexpected effects:

E.g., replacing x with *mother-of*(y) in the formula $\exists y.friends(x, y)$ you obtain the formula

$$\exists y.friends(mother\text{-}of(y), y)$$

Satisfiability and Validity

Definition (Model, satisfiability and validity)

An interpretation $\mathcal{I}$ is a model of ϕ under the assignment a, if

$$\mathcal{I} \models \phi[a]$$

A formula ϕ is satisfiable if there is some $\mathcal{I}$ and some assignment a such that $\mathcal{I} \models \phi[a]$.

A formula ϕ is unsatisfiable if it is not satisfiable.

A formula ϕ is valid if every $\mathcal{I}$ and every assignment a $\mathcal{I} \models \phi[a]$

Definition (Logical Consequence)

A formula ϕ is a logical consequence of a set of formulas Γ, in symbols $\Gamma \models \phi$, if for all interpretations $\mathcal{I}$ and for all assignment a

$$\mathcal{I} \models \Gamma[a] \quad \Longrightarrow \quad \mathcal{I} \models \phi[a]$$

where $\mathcal{I} \models \Gamma[a]$ means that $\mathcal{I}$ satisfies all the formulas in Γ under a.

Logical Consequence and reasoning

The notion of logical consequence enables us to determine if
"Mary is mortal" is a consequence of the facts that "Mary is a
person" and "All persons are mortal".
What we need to do is to determine if

$$Person(mary), \forall x Person(x) \supset Mortal(x) \models Mortal(mary)$$

Expressing properties in FOL

What is the meaning of the following FOL formulas?

1. $bought(Frank, dvd)$
2. $\exists x.bought(Frank, x)$
3. $\forall x.(bought(Frank, x) \rightarrow bought(Susan, x))$
4. $(\forall x.bought(Frank, x)) \rightarrow (\forall x.bought(Susan, x))$
5. $\forall x \exists y.bought(x, y)$
6. $\exists x \forall y.bought(x, y)$

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1. "Frank bought a dvd."
2. "Frank bought something."
3. "Susan bought everything that Frank bought."
4. "If Frank bought everything, so did Susan."
5. "Everyone bought something."
6. "Someone bought everything."

Expressing properties in FOL

Define an appropriate language and formalize the following sentences using FOL formulas.

1. All Students are smart.
2. There exists a student.
3. There exists a smart student.
4. Every student loves some student.
5. Every student loves some other student.
6. There is a student who is loved by every other student.
7. Bill is a student.
8. Bill takes either Analysis or Geometry (but not both).
9. Bill takes Analysis and Geometry.
10. Bill doesn't take Analysis.
11. No students love Bill.

Expressing properties in FOL

1. $\forall x.(Student(x) \rightarrow Smart(x))$

2. $\exists x.Student(x)$

3. $\exists x.(Student(x) \wedge Smart(x))$

4. $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge Loves(x, y)))$

5. $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge \neg(x = y) \wedge Loves(x, y)))$

6. $\exists x.(Student(x) \wedge \forall y.(Student(y) \wedge \neg(x = y) \rightarrow Loves(y, x)))$

7. $Student(Bill)$

8. $Takes(Bill, Analysis) \leftrightarrow \neg Takes(Bill, Geometry)$

9. $Takes(Bill, Analysis) \wedge Takes(Bill, Geometry)$

10. $\neg Takes(Bill, Analysis)$

11. $\neg\exists x.(Student(x) \wedge Loves(x, Bill))$

Expressing properties in FOL

For each property write a formula expressing the property, and for
each formula writhe the property it formalises.

- Every Man is Mortal

- Every Dog has a Tail

- There are two dogs

- Not every dog is white

- $\exists x.Dog(x) \wedge \exists y.Dog(y)$

- $\forall x, y(Dog(x) \wedge Dog(y) \supset x = y)$

Expressing properties in FOL

For each property write a formula expressing the property, and for each formula writhe the property it formalises.

- Every Man is Mortal
  $\forall x.Man(x) \supset Mortal(x)$

- Every Dog has a Tail
  $\forall x.Dog(x) \supset \exists y(PartOf(x,y) \land Tail(y))$

- There are two dogs
  $\exists x,y(Dog(x) \land Dog(y) \land x \neq y)$

- Not every dog is white
  $\neg\forall x.Dog(x) \supset White(x)$

- $\exists x.Dog(x) \land \exists y.Dog(y)$
  There is a dog

- $\forall x,y(Dog(x) \land Dog(y) \supset x = y)$
  There is at most one dog

First order theories

- Mathematics focuses on the study of properties of certain structures. E.g. Natural/Rational/Real/Complex numbers, Algebras, Monoids, Lattices, Partially-ordered sets, Topological spaces, fields, ...
- In knowledge representation, mathematical structures can be used as a reference abstract model for a real world feature. e.g.,
  - natural/rational/real numbers can be used to represent linear time;
  - trees can be used to represent possible future evolutions;
  - graphs can be used to represent maps;
  - ...
- Logics provides a rigorous way to describe certain classes of mathematical structures.

First order theory

Definition (First order theory)

A first order theory is a set of formulas of the FOL language closed under the logical consequence relation. That is, T is a theory iff $T \models A$ implies that $A \in T$

Remark

A FOL theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas (which are infinite).

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all $\phi \in T$, $\Omega \models \phi$.

First order theory (cont'd)

Definition

Finitely axiomatizable theory A theory T is finitely axiomatizable if it has a finite set of axioms.

Definition (Axiomatizable structure)

Given a class of mathematical structures C for a language L, we say that a theory T is a sound and complete axiomatization of C if and only if

$$T \models \phi \qquad \Longleftrightarrow \qquad \mathcal{I} \models \phi \ \text{ for all } \mathcal{I} \in C$$

Number theory (or Peano Arithmetic) *PA* $\mathcal{L}$ contains the constant symbol 0, the 1-nary function symbol *s*, (for successor) and two 2-nary function symbol $+$ and $*$

1. $0 \neq s(x)$
2. $s(x) = s(y) \supset x = y$
3. $x + 0 = x$
4. $x + s(y) = s(x + y)$
5. $x * 0 = 0$
6. $x * s(y) = (x * y) + x$
7. the Induction axiom schema: $\phi(0) \wedge \forall x.(\phi(x) \supset \phi(s(x))) \supset \forall x.\phi(x)$, for every formula $\phi(x)$ with at least one free variable

K. Gödel 1931 It's false that $\mathcal{I} \models PA$ if and only if $\mathcal{I}$ is isomorphic to the standard models for natural numbers.